

# On harmonic functions and the linear-growth case of Gromov's theorem

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## Abstract

We show that the space of harmonic functions on a finitely generated infinite group  $G$  is finite dimensional if, and only if,  $G$  has a finite-index subgroup isomorphic to  $\mathbb{Z}$ . A key tool is Wilkie and van den Dries's quantitative version of the linear-growth case of Gromov's theorem on groups of polynomial growth.

## 1 Introduction

Let  $G$  be a group with a finite symmetric generating set  $S$ , and for  $n = 1, 2, 3, \dots$  write

$$S^n = \{s_1 \cdots s_n : s_1, \dots, s_n \in S\}.$$

If there exist constants  $C$  and  $d$  such that for every  $n$  we have

$$|S^n| \leq Cn^d \tag{1.1}$$

then  $G$  is said to have *polynomial growth of degree  $d$  with respect to  $S$* . In fact, it is elementarily the case that this definition is independent of the choice of generating set, and so one can drop reference to the set  $S$  and say simply that  $G$  has *polynomial growth of degree  $d$* .

A famous and deep theorem of Gromov [1] states that a finitely generated group  $G$  has polynomial growth if, and only if, it is *virtually nilpotent*, which is to say it contains a nilpotent subgroup of finite index. This result has numerous applications in a variety of mathematical fields; see, for example, those listed in [3, pp. 10-12].

The conclusion of Gromov's theorem is particularly precise in the case where  $G$  is assumed to have *subquadratic growth*, which is to say when  $|S^n| \leq o(n^2)$ . It is essentially an exercise to verify that a finitely generated nilpotent group of subquadratic growth must contain  $\mathbb{Z}$  as a finite-index subgroup, and that in that case the group in fact exhibits *linear growth*, which is to say it satisfies (1.1) with  $d = 1$ . Thus the subquadratic case of Gromov's theorem implies that the following conditions are equivalent for a finitely generated infinite group  $G$ .

1.  $G$  has linear growth.

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2.  $G$  has subquadratic growth.
3.  $G$  contains  $\mathbb{Z}$  as a finite-index subgroup.

The non-trivial content of this equivalence is of course the implication (2)  $\implies$  (3). The full strength of Gromov's theorem is in fact not required to prove this implication; indeed, Wilkie and van den Dries [4] gave an elementary proof of it, with the index of  $\mathbb{Z}$  in  $G$  bounded in terms of the cardinality of the sets  $S^m \setminus S^{m-1}$ . We state their result precisely as Proposition 3.1 below.

The purpose of this note is to add to this list of equivalent conditions, as follows.

4. There exists a generating set for  $G$  with respect to which the space of harmonic functions on  $G$  is finite dimensional.
5. The space of harmonic functions on  $G$  is finite dimensional with respect to every generating set for  $G$ .

Here, as is customary, a function  $f : G \rightarrow \mathbb{R}$  is said to be *harmonic* with respect to a generating set  $S$  for  $G$  if

$$f(x) = \frac{1}{|S|} \sum_{s \in S} f(xs)$$

for every  $x \in G$ . Specifically, we prove the following theorem.

**Theorem 1.** *Let  $G$  be an infinite group with finite symmetric generating set  $S$ . Then the space of functions on  $G$  that are harmonic with respect to  $S$  is finite dimensional if, and only if,  $G$  contains a finite-index subgroup isomorphic to  $\mathbb{Z}$ .*

Let us remark at this point that it is not surprising that finiteness of the dimension of the space of harmonic functions should be related to the growth of a group. Kleiner [2] has shown that a group  $G$  of polynomial growth has a finite-dimensional space of *Lipschitz* harmonic functions with respect to an arbitrary generating set, where a function  $f : G \rightarrow \mathbb{R}$  is said to be *Lipschitz* with respect to a generating set  $S$  if there exists a constant  $C$  such that  $|f(x) - f(xs)| \leq C$  for every  $x \in G, s \in S$ . He was then able to deduce Gromov's theorem from this fact.

Theorem 1 essentially contains two statements. The 'direct' statement is that a group containing  $\mathbb{Z}$  as a finite-index subgroup has a finite-dimensional space of harmonic functions. This is not difficult, but for completeness we present it in Section 2. The more interesting 'inverse' statement is that a finitely generated infinite group with a finite-dimensional space of harmonic functions must contain  $\mathbb{Z}$  as a finite-index subgroup. We prove this in Section 3. The key tool is Wilkie and van den Dries's quantitative version of the subquadratic case of Gromov's theorem, which we state in the same section.

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## 2 Proof of the direct statement

It will be convenient to make the following definitions.

**Definition 2.1** (Boundary and interior of a set). Given a subset  $X$  of a group  $G$  with a symmetric generating set  $S$ , define the *interior* of  $X$  (with respect to  $S$ ) to be the set

$$X^\circ := \{x \in X : xS \subset X\},$$

and the *boundary* of  $X$  to be the set

$$\partial X := X \setminus X^\circ.$$

**Definition 2.2** (Locally harmonic). Let  $G$  be a group with finite symmetric generating set  $S$  and let  $X$  be a subset of  $G$ . We shall say that a function  $f : X \rightarrow \mathbb{R}$  is *locally harmonic* if for each  $x \in X$  with  $xS \subset X$  we have  $f(x) = \mathbb{E}_{s \in S} xs$ .

The following is an immediate consequence of the definition of harmonicity.

**Lemma 2.3** (Maximum principle). *Let  $G$  be a group and let  $X$  be a finite subset of  $G$ . Suppose that  $f : X \rightarrow \mathbb{R}$  is locally harmonic. Then the maximum value of  $X$  is attained on the boundary of  $X$ .*

The maximum principle in turn yields the following familiar property of harmonic functions.

**Corollary 2.4** (Locally harmonic functions are determined by their boundary values). *Let  $G$  be a group and let  $X$  be a finite subset of  $G$ . Suppose that  $f, f' : X \rightarrow \mathbb{R}$  are locally harmonic and that  $f|_{\partial X} = f'|_{\partial X}$ . Then  $f = f'$ .*

*Proof.* The maximum principle implies that both  $f - f'$  and  $f' - f$  have zero as their maximum value.  $\square$

We are now in a position to prove the direct statement of Theorem 1. Let  $G$  be a group containing  $\mathbb{Z}$  as a finite-index subgroup. Let  $T$  be a right transversal of  $\mathbb{Z}$ , which is to say a set such that every element  $x \in G$  can be written uniquely in the form  $\zeta(x)\tau(x)$  with  $\zeta(x) \in \mathbb{Z}$  and  $\tau(x) \in T$ . The set  $T$  has cardinality  $[G : \mathbb{Z}]$ , and so in particular we may define  $M$  to be the maximum over the (finite) set  $\{|\zeta(ts)| : s \in S, t \in T\}$ .

Now for every  $n \in \mathbb{Z}, t \in T, s \in S$  we have  $nts = n\zeta(ts)\tau(ts) \in [n - M, n + M]T$ , and so in particular for  $N \in \mathbb{Z}$  we have  $[-N, N]TS \subset [-M - N, M + N]T$ . It follows that  $[-N, N]T$  is contained in the interior of  $[-M - N, M + N]T$ , and so the boundary of  $[-M - N, M + N]T$  has cardinality at most  $2M|T|$ . Corollary 2.4 therefore implies that the space of locally harmonic functions on  $[-M - N, M + N]T$  is of dimension at most  $2M|T|$ . However,  $G = \bigcup_{N=1}^{\infty} [-M - N, M + N]T$ , and so the space of harmonic functions on  $G$  is also of dimension at most  $2M|T|$ .  $\square$

*Remark 2.5.* Setting  $G = \mathbb{Z}$  and  $S = [-M, M]$  shows that the bound  $2M|T|$  on the dimension of the space of harmonic functions is best possible. In particular, the dimension depends on the generating set  $S$  as well as on the group  $G$ .

### 3 Proof of the inverse statement

The inverse statement of Theorem 1 is an immediate consequence of the following two results.

**Proposition 3.1** (Wilkie & van den Dries [4]). *Let  $G$  be an infinite group generated by a finite symmetric set  $S$ . Let  $m > 0$  and suppose that  $|S^m \setminus S^{m-1}| \leq m$ . Then  $G$  contains  $\mathbb{Z}$  as a subgroup of index at most  $m^4$ .*

**Lemma 3.2.** *Let  $G$  be a group generated by a finite symmetric set  $S$  and suppose that  $|S^m \setminus S^{m-1}| > m$  for every  $m \in \mathbb{N}$ . Then the space of harmonic functions on  $G$  with respect to  $S$  is infinite dimensional.*

Noting that  $f$  is harmonic on  $G$  if and only if each restriction of  $f$  to a ball  $S^k$  is locally harmonic, the following result implies Lemma 3.2 and hence the inverse direction of Theorem 1.

**Lemma 3.3.** *Let  $G$  be a group with finite symmetric generating set  $S$  and suppose that, for every  $m \in \mathbb{N}$ , we have*

$$|S^m \setminus S^{m-1}| > m. \quad (3.1)$$

*Then there exists an increasing sequence  $m_1, m_2, \dots$  of natural numbers with the property that whenever  $f : S^{m_{n-1}} \rightarrow \mathbb{R}$  is locally harmonic there exists an element  $g \in S^{m_n}$  such that for every  $\lambda \in \mathbb{R}$  there is a locally harmonic function  $f' : S^{m_n} \rightarrow \mathbb{R}$  that extends  $f$  and has  $f'(g) = \lambda$ .*

*Proof.* We define the sequence  $m_1, m_2, \dots$  recursively. Let  $m_1 = 1$ , and suppose that  $m_1, \dots, m_{n-1}$  are all defined and satisfy the required property. Then set

$$m_n = |S| |S^{m_{n-1}}|. \quad (3.2)$$

Now write  $F_n$  for the space of functions  $S^{m_n} \rightarrow \mathbb{R}$ . Given  $f : S^{m_{n-1}} \rightarrow \mathbb{R}$  the set of locally harmonic functions on  $S^{m_n}$  that extend  $f$  is precisely the set of functions  $f' \in F_n$  satisfying the linear conditions

$$f'(x) = f(x) \quad (x \in S^{m_{n-1}}) \quad (3.3)$$

$$f'(x) = \mathbb{E}_{s \in S} f'(xs) \quad (xS \subset S^{m_n}) \quad (3.4)$$

The dimension of  $F_n$  is  $|S^{m_n}|$ , and so it will be sufficient to show that the number of conditions of the forms (3.3) and (3.4) is less than  $|S^{m_n}|$ .

There are precisely  $|S^{m_{n-1}}|$  conditions of the form (3.3), whilst the number of conditions of the form (3.4) is equal to the number of elements  $x \in S^{m_n}$  for which  $xS \subset S^{m_n}$ . However, (3.1) and (3.2) imply that  $|S^{m_{n+1}} \setminus S^{m_n}| > |S| |S^{m_{n-1}}|$ , and since each  $x \in G$  has only  $|S|$  neighbours this implies that there are more than  $|S^{m_{n-1}}|$  elements  $x \in S^{m_n}$  with at least one neighbour lying outside  $S^{m_n}$ .

Put another way, there are more than  $|S^{m_{n-1}}|$  elements  $x \in S^{m_n}$  such that  $xS$  is not contained within  $S^{m_n}$  and that therefore do not impose a condition of the form (3.4). The total number of conditions of the forms (3.3) and (3.4) is therefore less than  $|S^{m_n}|$ , and the lemma is proved.  $\square$

## References

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